THE GENERALIZED TELEPARALLEL STRUCTURE.

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The teleparallel geometrical structures has evoked in resent time a considerable interest for various reasons. It is considered as a possible physical relevant geometry itself as well as an essential part of generalized non-Riemannian theories such as the metric - affine gravity. Another important subject is the various applications of the frame technique in physical theories based on classical (pseudo) Riemannian geometry.

Teleparallel structure exists on a manifold of a vanishing second Stiefel-Whitney class. This is a restrictive condition relative to existence of the Lorentzian metrics. It guarantees:

1) orientability, time- and space-orientability,
2) existence of a unique Riemannian and a unique Lorentzian structures,
3) existence of a spinorial structure,
4) good posing of the Cauchy problem.

Thus the restriction of the space-time topology is good motivated by physical requirements.

The teleparallel structures actually used in gravity are described by an *equivalence class* $[\vartheta^a]$ of coframe fields on a differential manifold. The *equivalence relation* on this class is constructed by a certain group of transformations. The group $G = GL(n, \mathbb{R})$ provides the maximal equivalence class: it includes all the possible coframes at a point. Certainly, a sufficient large subgroup of $GL(n, \mathbb{R})$ can also be chosen for the equivalence relation. Thus an important question for the teleparallel construction is:

What subgroups of the general linear group produce relevant physical models?

The choice of a subgroup of $GL(n, \mathbb{R})$ can be described by declaration of a set of invariant conditions on a matrix $G^a{}_b$. These invariants are directly connected with geometrical and physical objects.

The choice $G = SL(n, \mathbb{R})$ provides the maximal equivalence class which preserves the volume element structure on M.

Similarly, the choice $G = SO(n, \mathbb{R})$ provides the maximal equivalence class which preserves the Riemannian metric structure on M.

As for the Lorentzian structure it is preserved by the action of the group $G = SO(1, n-1, \mathbb{R})$ or of a suitable subgroup of it.

All the structures above are naturally constructed by coframes.

The symplectic structure can also be constructed via the coframe field.

$$w = w_{ab}\vartheta^a \wedge \vartheta^b, \qquad w_{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1}$$

This structure is preserved by $Sp(2n, \mathbb{R})$ transformations of the coframe field. The geometry of a symplectic manifold can be developed in the form similar to the

Riemannian geometry. It means that a torsion free connection preserving the symplectic metric can be introduced. This is a symplectic analog of the Levi-Civita connection.
It is not sufficient to have a symplectic structure on the physical manifold because the lengths of the vectors have also to be calculated. Thus it is interesting to consider a mixed structure with two different metrics: one pseudo (Riemannian) and the second symplectic. Note that such structure is in a some sense equivalent to the teleparallel structure. Indeed, the coframe in a general coordinate chart has n^2 independent components. As for a pseudo(Riemannian) metric it incorporates $\frac{n(n+1)}{2}$ independent products of the coframe components. The symplectic metric furnishes the rest, $\frac{n(n-1)}{2}$ independent products. Thus a coframe structure can be considered as a combination of a metric $\{g\}$ and a symplectic form $\{w\}$. Since $\{g\}$ is reserved for gravity it is plausible to define the electromagnetic field by a symplectic form.
In order to preserve the (pseudo)Riemannian and the symplectic metrics under global transformations of the coframes one has to consider the intersection of the corresponding groups.
In the case of a manifold with Euclidean signature the resulting group is well known

$$O(n,\mathbb{R}) \cap Sp(n,\mathbb{R}) = U(\frac{n}{2}).$$

For a 4D manifold it turns to be the basic group of the electro-week interaction

$$O(4,\mathbb{R}) \cap Sp(4,\mathbb{R}) = U(1) \times SU(2).$$

In the case of a manifold with Lorentzian signature the resulting group

$$O(1,n-1,\mathbb{R}) \cap Sp(n,\mathbb{R})$$

involves the boosts. It is non-compact, consequently, it is interesting to look for a maximal compact sub-group of it.
By introducing of a 3-indexed of the torsion $C^a{}_{bc} = e_c \rfloor e_b \rfloor d\vartheta^a$ the general (pseudo)Riemannian quadratic Lagrangian is represented. This Lagrangian is a linear combination of 3 independent pieces.

$$\mathcal{L}_{RT} = \Big(\alpha_1 C_{abc}C^{abc} + \alpha_2 C_{abc}C^{bac} + \alpha_3 C_a C^a\Big) * 1 \qquad (2)$$

It coincides with the translation invariant Lagrangian of Rumpf.
The quadratic teleparallel Lagrangian for a teleparallel manifold with a symplectic structure is unique

$$\mathcal{L}_{symp} = w^{nk} C^m{}_{ma} C^a{}_{nk} * 1.$$

As for a quadratic Riemannian-symplectic teleparallel Lagrangian it is a linear combination of 16 independent 4-forms, which are constructed explicitly.
The field equations for the structures described above will be studied.